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Unitary irreducible representations of covariant q-oscillators

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Abstract. Unitary irreducible representations of n independent q-oscillators are used for the construction of all unitary irreducible representations of the $SU_q(n)$-covariant system of q-oscillators.

The most important applications of q-oscillators are related to the construction of representations of q-deformed Lie algebras [1–5]. One of the simplest forms in which the q-oscillator algebra usually appears is

$$bb^+ - q^2 b^+ b = 1 \tag{1}$$

where b and b^+ are the annihilation and creation operators respectively, and q is the deformation parameter, which in what follows is assumed to be a positive real number, $q > 0$.

In [6] it was shown that the unitary irreducible representations of the algebra (1) can be classified according to the sign of the definite operator

$$K = [b, b^+]. \tag{2}$$

Namely, $K > 0$ corresponds to the case of Fock representations of the algebra (1) with a non-degenerate spectrum of K given for any $q > 0$ by q^{2k}, $k = 0, 1, 2, \ldots$; $K < 0$ corresponds to non-Fock representations for $0 < q < 1$ with the non-degenerate spectrum of K being $-q^{2(k+\gamma)}$, $k \in \mathbb{Z}$ and $\gamma \in (0, 1)$. Finally, $K = 0$ corresponds to a degenerate representation for $0 < q < 1$ with $b^+b = bb^+ = (1 - q^2)^{-1} I$. Let us mention that there is also an infinite-dimensional degenerate representation ($b|n\rangle = (1 - q^2)^{-1}|n - 1\rangle$, $b^+|n\rangle = (1 - q^2)^{-1}|n + 1\rangle$, $n \in \mathbb{Z}$) and a one-dimensional trivial representation ($b = b^+ = (1 - q^2)^{-1} I$). We shall not use the latter representations below.

The generalization of the above statements to the case of n pairs of independent q-oscillators b_i, b_i^+, $i = 1, \ldots, n$ satisfying the relations

$$\begin{aligned} &b_i b_j^+ - q^{2\delta_{ij}} b_j^+ b_i = \delta_{ij} \qquad q > 0 \\ &[b_i, b_j] = [b_i^+, b_j^+] = 0 \end{aligned} \tag{3}$$

is straightforward. The representations of (3) will be given as the tensor product of the representations of each pair of q-oscillators entering into (3).

First we notice that in any unitary irreducible (infinite-dimensional) representation there exist operators $N_i = N_i^+$ and real parameters λ_i, $i = 1, \dots, n$ (specified below) that satisfy

$$\begin{aligned} &[N_i, N_j] = 0 \\ &[N_i, b_j] = [N_i, b_j^+] = 0 \qquad i \neq j \\ &[N_i, b_i] = -b_i \qquad [N_i, b_i^+] = b_i^+ \end{aligned} \tag{4}$$

and

$$b_i^+ b_i = \frac{1 + \lambda_i q^{2N_i}}{1 - q^2} \qquad b_i b_i^+ = \frac{1 + \lambda_i q^{2N_i+2}}{1 - q^2}. \tag{5}$$

The admissible λ_i and N_i in (4) and (5) can be classified into two classes [6]:

(A) Fock representations when $\lambda_i = -1$ and the eigenvalues of N_i are $n_i = 0, 1, 2, \dots$; in this case all the values of $q > 0$ are allowed; and

(B) non-Fock representations when $\lambda_i \geqslant 0$ and the eigenvalues of N_i are all the integers n_i; in this case only $0 < q < 1$ is allowed.

Let us note that the non-Fock representations of two pairs of q-oscillators, b, b^+ and $\tilde{b}, \tilde{b}^+$ satisfying

$$b^+ b = \frac{1 + \lambda q^{2N}}{1 - q^2} \qquad \tilde{b}^+ \tilde{b} = \frac{1 + \tilde{\lambda} q^{2\tilde{N}}}{1 - q^2}$$

are unitarily equivalent if $\tilde{\lambda} = q^k \lambda$ with k being an integer number. Furthermore, by putting

$$b_0 = (1 + \lambda q^{2N+2})^{-1/2} b \qquad b_0^+ = b^+ (1 + \lambda q^{2N+2})^{-1/2}$$

we obtain a non-Fock pair of q-oscillators which satisfy the relations

$$b_0 b_0^+ = b_0^+ b_0 = \frac{1}{1 - q^2}$$

and which thus corresponds to the degenerate case $\lambda_0 = 0$. We note, however, that the mapping $b \to b_0$, $b^+ \to b_0^+$ is not unitary.

In this paper we shall use the unitary irreducible representations of the algebra (3) to construct all the irreducible representations of the $SU_q(n)$-covariant system of q-oscillators. A general classification of the latter representations was given in [7] by using first-order differential calculus on a q-plane. However, as we shall see below, the construction of such representations in terms of q-oscillators can be performed in the simplest form. In addition, we believe that the approach presented here can also be used for the realization of more general systems such as a supercovariant system of q-oscillators [8].

Let a_i, a_i^+, $i = 1, \dots, n$ be a system of q-oscillators satisfying the $SU_q(n)$-covariant algebra:

$$\begin{aligned} &a_j a_i = q a_i a_j \qquad a_i^+ a_j^+ = q a_j^+ a_i^+ \qquad i < j \\ &a_j a_i^+ = q a_i^+ a_j \qquad i \neq j \\ &a_i a_i^+ - q^2 a_i^+ a_i = \Lambda_{i+1} \end{aligned} \tag{6}$$

where $0 < q < 1$, and

$$\begin{aligned} &\Lambda_i = 1 - (1 - q^2) \sum_{k \geqslant i}^{n} a_k^+ a_k \qquad i = 1, \dots, n \\ &\Lambda_{n+1} = 1. \end{aligned} \tag{7}$$

From equations (6) and (7) the following basic relations for the operators Λ_j can be deduced:

$$\begin{aligned}\Lambda_{j+1} - \Lambda_j &= (1-q^2)a_j^+ a_j\\ \Lambda_{j+1} - q^2\Lambda_j &= (1-q^2)a_j a_j^+\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\Lambda_j a_k^+ = a_k^+\Lambda_j \qquad a_k\Lambda_j = \Lambda_j a_k \qquad j > k\\ &\Lambda_j a_k^+ = q^2 a_k^+\Lambda_j \qquad a_k\Lambda_j = q^2\Lambda_j a_k \qquad j \leqslant k.\end{aligned} \tag{9}$$

To construct the irreducible representations of (6) and (7) we shall consider four cases.

(i) Assume that $\Lambda_{i+1} = R_{i+1}^2 > 0$ and let us define

$$a_i = F_i(N_i, \ldots)b_i \qquad a_i^+ = b_i^+ F_i(N_i, \ldots) \tag{10}$$

where b_i, b_i^+ satisfy the q-oscillator algebra (3) and the N_i satisfy the relations (4). The dots in (10) indicate the dependence of the functions F_i on N_j, $j \neq 1$ and hereafter will be omitted. Then according to (4) and (5)

$$\begin{aligned}a_i a_i^+ &= \frac{1+\lambda_i q^{2N_i+2}}{1-q^2}F_i^2(N_i)\\ a_i^+ a_i &= \frac{1+\lambda_i q^{2N_i}}{1-q^2}F_i^2(N_i-1)\end{aligned} \tag{11}$$

where the values of λ_i and N_i are determined in correspondence with the representation (Fock or non-Fock) chosen for the q-oscillators b_i (see statements (A) and (B) after equation (5)).

From relations (9) and definitions (10), it is easy to conclude that

$$R_{i+1} = R_{i+1}(N_{i+1}, \ldots, N_n). \tag{12}$$

Indeed, since the operator Λ_{i+1} commutes with a_j and a_j^+ for $j = 1, \ldots, i$ (cf equations (9)), then it cannot depend on N_j. Therefore, $\Lambda_{i+1} = \Lambda_{i+1}\ (N_{i+1}, \ldots, N_n)$. Let us now assume that the q-oscillators b_i are given in the Fock representation, i.e. $\lambda_i = -1$. Then substituting relations (11) into the last equation in (6) we obtain the recurrence relation

$$\frac{1-q^{2N_i+2}}{1-q^2}F_i^2(N_i) - q^2\frac{1-q^{2N_i}}{1-q^2}F_i^2(N_i-1) = \Lambda_{i+1} \tag{13}$$

which has the general solution

$$F_i^2(N_i) = \Lambda_{i+1} + \frac{cq^{2N_i+1}}{1-q^{2N_i+2}} \tag{14}$$

where c is a constant.

In particular, for $c = 0$ we obtain the solution

$$F_i = R_{i+1}(N_{i+1}, \ldots, N_n). \tag{15}$$

Recalling that $\Lambda_i = R_i^2 > 0$, we obtain from (8), (11) and (15) the recurrence relation

$$R_i = q^{N_i}R_{i+1}. \tag{16}$$

Since by definition $R_{n+1} = 1$, finally we have the solution

$$F_i = R_{i+1} = q^{N_{i+1}+\ldots+N_n} \tag{17}$$

found in [8] provided that $\Lambda_i > 0$ for $i = 1, \ldots, n$.

(ii) Let us now consider the case $\Lambda_{i+1} = -R_{i+1}^2 < 0$ and define

$$a_i = b_i^+ G_i(N_i) \qquad a_i^+ = G_i(N_i)b_i \tag{18}$$

where b_i, b_i^+ satisfy relations (3)–(5) and the dependence of the functions G_i on N_j, $j \neq i$, $j \neq 1$ has been omitted.

Then from (5) it follows that

$$\begin{aligned} a_i a_i^+ &= \frac{1+\lambda_i q^{2N_i}}{1-q^2} G_i^2(N_i - 1) \\ a_i^+ a_i &= \frac{1+\lambda_i q^{2N_i+2}}{1-q^2} G_i^2(N_i). \end{aligned} \tag{19}$$

Assuming that the operators b_i, b_i^+ are Fock oscillators (i.e. $\lambda_i = -1$) and substituting (19) into (6) we arrive at the relation

$$\frac{1-q^{2N_i}}{1-q^2}\, G_i^2(N_i - 1) - q^2\, \frac{1-q^{2N_i+2}}{1-q^2}\, G_i^2(N_i) = \Lambda_{i+1} \tag{20}$$

with $\Lambda_{i+1} = \Lambda_{i+1}(N_{i+1}, \ldots, N_n)$.

The general solution of (20) is given by

$$G_i^2(N_i) = -q^{-2N_i-2}\Lambda_{i+1} + \frac{cq^{-2N_i}}{1-q^{2N_i+2}} \tag{21}$$

where c is an arbitrary constant.

Since $\Lambda_{i+1} = -R_{i+1}^2 < 0$, for $c = 0$ we have

$$G_i = q^{-N_i-1} R_{i+1}(N_{i+1}, \ldots, N_n). \tag{22}$$

Furthermore, from relations (6) we obtain the recurrence relation

$$R_i = q^{-N_i-1} R_{i+1}. \tag{23}$$

If $\Lambda_i < 0$ for $i = 1, \ldots, p < n$, then from (22) we find

$$G_i = q^{-(N_i+1)-\cdots-(N_{p-1}+1)} R_p(N_p, \ldots, N_n) \tag{24}$$

valid for $i = 1, \ldots, p-1$, where R_p is determined by $\Lambda_p = -R_p^2$.

(iii) Let us assume that $\Lambda_{i+1} = R_{i+1}^2 > 0$ and that the q-oscillators b_i are given in the non-Fock representation:

$$b_i^+ b_i = \frac{1+\varepsilon^2 q^{2N_i}}{1-q^2} \qquad b_i b_i^+ = \frac{1+\varepsilon^2 q^{2N_i+2}}{1-q^2} \tag{25}$$

where $\lambda_i \equiv \varepsilon^2 \geqslant 0$.

Defining a_i, a_i^+ as in (10) and repeating all the steps from case (i) we find $F_i = R_{i+1}$. Substituting a_i and a_i^+ into equation (8) and using (25) we obtain

$$\Lambda_i = -\varepsilon^2 q^{2N_i} \Lambda_{i+1} \leqslant 0. \tag{26}$$

Thus, we conclude that the non-Fock q-oscillators b_i, b_i^+ allow us to switch from pattern (i) to pattern (ii) provided that $\varepsilon \neq 0$. If $\varepsilon = 0$, then we arrive at $\Lambda_i = 0$.

(iv) Assume that $\Lambda_{i+1} = 0$. Then we can take b_i, b_i^+ in the non-Fock representation with $\lambda_i = 0$ satisfying

$$b_i^+ b_i = b_i b_i^+ = \frac{1}{1-q^2}. \tag{27}$$

Defining $a_i = F_i(N_i) b_i$, $a_i^+ = b_i^+ F_i(N_i)$, we obtain from (6) the equation

$$F_i^2(N_i) - q^2 F_i^2(N_{i-1}) = 0 \tag{28}$$

which has the solution

$$F_i = \xi(N_{i+1}, \ldots, N_n) q^{N_i}. \tag{29}$$

Using $\Lambda_{i+1} = 0$, finally we find from equation (8)

$$\Lambda_i = -q^{-2} F_i^2 = -\xi^2 q^{2N_i - 2} < 0. \tag{30}$$

Therefore we see that this case leads again to case (ii).

Thus, from the four cases considered above we conclude that the most general pattern for the classification of all irreducible representations of the $SU_q(n)$-covariant system of q-oscillators defined by (6) is the following.

Let b_i, b_i^+, $i = m+1, \ldots, n$ be Fock oscillators and define a_i, a_i^+ as in (10). Then according to case (i), $\Lambda_{i+1} = q^{2(N_{i+1}+\ldots+N_n)} > 0$ for $i = m+1, \ldots, n-1$ and $\Lambda_{n+1} = 1$ by definition.

Now if b_m is a degenerate non-Fock oscillator with $\varepsilon = 0$, then from case (iv) we have $\Lambda_{m+1} = 0$ and $\Lambda_m < 0$†. Therefore, the use of a degenerate oscillator b_m allows us to switch from positive to negative values of Λ_i and case (ii) implies that $\Lambda_i < 0$ for $i = m-1, \ldots, 1$ if b_i are Fock q-oscillators.

Finally if b_m is a non-Fock oscillator with $\varepsilon \neq 0$, then, according to case (iii), $\Lambda_{m+1} > 0$ and $\Lambda_m < 0$. Therefore we arrive again at case (ii), where for the Fock q-oscillators $b_1, b_2, \ldots, b_{m-1}$ one has $\Lambda_i < 0$, $i = 1, \ldots, m-1$.

We notice that the above construction exactly reproduces all the unitary irreducible representations of the $SU_q(n)$-covariant algebra (6) found in [7].

Let us now consider some simple illustrative examples of the use of the unitary irreducible representations of the q-oscillator algebra (3) for the construction of q-deformed algebras. We introduce the operators

$$a_i = q^{-N_i/2} b_i \qquad a_i^+ = b_i^+ q^{-N_i/2} \qquad i = 1, \ldots, n \tag{31}$$

where the operators b_i, b_i^+ satisfy relations (3). The definitions (31) lead to a system of n independent q-oscillators a_i, a_i^+ which satisfy the relations

$$\begin{aligned} &a_i a_i^+ - q a_i^+ a_i = q^{-N_i} \\ &[N_i, a_i] = -a_i \qquad [N_i, a_i^+] = a_i^+ \end{aligned} \tag{32}$$

where N_i is defined by $|K_i| = |[b_i, b_i^+]| = q^{2(N_i+\gamma_i)}$ and γ_i are real parameters specifying the q-oscillator representations in question ($\gamma_i = 0$ in the Fock case, whereas $\gamma_i \in (0, 1)$ in the non-Fock case).

We shall take for simplicity two independent pairs of q-oscillators a_i, a_i^+, $i = 1, 2$ given by (31) and (32) and construct the following set of operators

$$\begin{aligned} &J^+ = a_1^+ a_2 \qquad J^- = a_2^+ a_1 \\ &J^0 = \tfrac{1}{2}(N_1 - N_2). \end{aligned} \tag{33}$$

Now if we assume that both pairs in (31) are in the Fock representation, we obtain from (33) the $SU_q(2)$ algebra [1, 2]

$$[J^0, J^\pm] = \pm J^\pm \qquad [J^+, J^-] = [2J^0]_q \tag{34}$$

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$. The Casimir operator for (34) is given as

$$C = \left[J^0 + \tfrac{1}{2}\right]_q^2 + J^- J^+ = \left[\tfrac{1}{2}(N_1 + N_2 + 1)\right]_q^2. \tag{35}$$

† In principle one can add some $\Lambda_i = 0$, $i = m, \ldots, p+1$ $(m > p)$ and then, using b_p as a degenerate non-Fock q-oscillator, switch to the case $\Lambda_p < 0$.

Restricting the Fock space

$$F = \{|n_1, n_2\rangle;\ n_1, n_2 = 0, 1, 2, \ldots\}$$

to invariant subspaces given by $n_1+n_2+1 = n$ with n being constant, we obtain the unitary irreducible representations of $SU_q(2)$.

When both pairs of operators in (31) are given in the non-Fock representation, then from (33) we obtain the $SU_q(1,1)$ algebra [9]

$$[J^0, J^{\pm}] = \pm J^{\pm} \qquad [J^+, J^-] = -[2J^0]_q \tag{36}$$

with the Casimir operator

$$C = \left[J^0 + \tfrac{1}{2}\right]_q^2 - J^- J^+ = -\left\{\tfrac{1}{2}(N_1 + N_2 + \gamma_1 + \gamma_2 + 1)\right\}_q^2 \tag{37}$$

where $\{x\}_q \equiv (q^x + q^{-x})/(q - q^{-1})$. In this case the irreducible representations are again specified by the condition $n_1 + n_2 + 1 = n$ with n being constant.

If instead of (33) we use the operators

$$\begin{aligned} &J^+ = a_1^+ a_2^+ \qquad J^- = a_2 a_1 \\ &J^0 = \tfrac{1}{2}(N_1 + N_2 + 1) \end{aligned} \tag{38}$$

then one can conclude that when both pairs of q-oscillators are given in the Fock representation, the operators (38) satisfy the $SU_q(1,1)$ algebra (36) with the Casimir operator defined as

$$C = \left[\tfrac{1}{2}(N_1 - N_2)\right]_q^2 . \tag{39}$$

In the case of non-Fock q-oscillators, the same algebra (36) holds for the operators (38). In the latter case the Casimir operator reads as

$$C = -\left\{\tfrac{1}{2}(N_1 - N_2 + \gamma_1 - \gamma_2)\right\}_q^2 . \tag{40}$$

From these examples we see that the operators defined in (33) and (38) with Fock q-oscillators lead to unitary irreducible representations of $SU_q(2)$ and to a (discrete) series of representations of $SU_q(1,1)$ respectively, both with positive Casimir operators. These representations have in the limit $q \to 1$ canonical realizations in terms of standard oscillators. The representations (33) and (38) with non-Fock q-oscillators both lead to representations of $SU_q(1,1)$ from (principal) series with negative Casimir operators. We stress that the latter non-Fock representations have no classical analogies since they disappear for $q \to 1$ (e.g. $C \to -\infty$ in this limit, see equations (37) and (40)).

Finally let us mention that some other possible applications of q-oscillators may concern the construction of the representations of superunitary $SU_q(m|n)$ systems (the Fock representations in this case were found in [8]) and superalgebras like osp_q (1) and (2) (see e.g. [10] for the Fock case). We believe that the construction of the representations of q-deformed (super)algebras can also be performed in the convenient framework of q-oscillators along similar lines to the ones presented here.

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